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Sharp parameter ranges in the uniform anti-maximum principle for second-order ordinary differential operators

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To L.E. Payne on the occasion of his 80th birthday

Abstract. We consider the equation $(pu')' - qu + \lambda wu = f$ in $(0,1)$ subject to homogenous boundary conditions at $x = 0$ and $x = 1$, e.g., $u'(0) = u'(1) = 0$. Let λ_1 be the first eigenvalue of the corresponding Sturm-Liouville problem. If $f \leq 0$ but $\not\equiv 0$ then it is known that there exists $\delta > 0$ (independent on f) such that for $\lambda \in (\lambda_1, \lambda_1 + \delta]$ any solution u must be negative. This so-called *uniform anti-maximum principle (UAMP)* goes back to Clément, Peletier [4]. In this paper we establish the sharp values of δ for which (UAMP) holds. The same phenomenon, including sharp values of δ, can be shown for the radially symmetric p-Laplacian on balls and annuli in $\mathbb{R}^n$ provided $1 \leq n < p$. The results are illustrated by explicitly computed examples.

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1. Introduction and main results

Let $L = a_{ij}(x)\partial^2_{ij} + b_i(x)\partial_i + c(x)$ be a uniformly elliptic operator on a bounded C^2-domain $\Omega \subset \mathbb{R}^n$ with continuous coefficients. Consider the boundary value problem

$$Lu + \lambda u = f \text{ in } \Omega, \quad \partial_\nu u + bu = 0 \text{ on } \partial\Omega \tag{1}$$

with a C^1-function $b \geq 0$ and $f \in L^q(\Omega)$, $q > n$. Let λ_1 denote the first eigenvalue of L subject to the above boundary condition. Then two important principles hold for solutions $u \in W^{2,q}(\Omega)$ of (1):

Maximum principle (MP): If $f \leq 0$, $f < 0$ on a set of positive measure and $\lambda < \lambda_1$ then $u > 0$ in $\overline{\Omega}$.

Anti-maximum principle (AMP): If $f \leq 0$, $f < 0$ on a set of positive measure then there exists $\delta = \delta(f) > 0$ such that $\lambda \in (\lambda_1, \lambda_1 + \delta]$ implies $u < 0$ in Ω.

The (AMP) was discovered by Clément, Peletier [4]. In the same paper a proof of the well known (MP) is given. In [4] the authors also consider the *uniform anti-maximum principle (UAMP)*, where the constant δ does not depend on the data f. They showed that (UAMP) holds in dimension $n = 1$ (and does not hold in higher dimensions). For example, by computing the Green function $G(s,t)$ for the operator $d^2/dx^2 + \lambda$ on the interval $(0,1)$ with Neumann boundary conditions at $x = 0, 1$ one finds that $G(s,t) < 0$ for $\lambda \in (0, \pi^2/4]$, whereas $G(s,t)$ is sign-changing for $\lambda > \pi^2/4$. Hence for

$$u'' + \lambda u = f \text{ in } (0,1), \quad u'(0) = u'(1) = 0$$

(UAMP) holds precisely for $\lambda \in (0, \pi^2/4]$.

Subsequently, both (AMP) and (UAMP) have been extended to linear problems with sign-changing weight by Hess [9] and Godoy et al. [8]. Recently Clément and Sweers [5] found conditions under which (AMP) and (UAMP) hold for higher-order elliptic boundary value problems. For the p-Laplacian div $(|\nabla u|^{p-2}\nabla u)$ Fleckinger et al. [6] showed the (AMP) by a proof different to the one in [4]. Fleckinger and Takač [7] consider (AMP) for p-Laplacian equations, cooperative systems and even Schrödinger operators in $\mathbb{R}^n$. For the first time the question of the sharp constant in the (UAMP) for the p-Laplacian on a bounded domain in $\mathbb{R}^n$ with $1 \le n < p$ and homogeneous Neumann boundary conditions was addressed by Arias et al. [1]. Below we will compare our results with the ones in [1].

In this paper we characterize the precise parameter-range of (UAMP) for the problem

$$(pu')' - qu + \lambda wu = f \text{ in } (0,1), \quad Bu = 0, \tag{2}$$

where Bu is an abbreviation for the boundary condition

$$(\alpha u + pu')|_{x=0} = 0, \quad (\beta u + pu')|_{x=1} = 0$$

with constants $\alpha, \beta \in \mathbb{R}$. We assume $p, q, w, f \in C[0,1]$ and $p, w > 0$ in $[0,1]$. Solutions are understood such that $u, pu' \in C^1[0,1]$. To describe our results we use the following notation: let $\lambda_1^{\alpha\beta}$ be the first eigenvalue of the Sturm-Liouville problem associated with (2). In this notation $\lambda_1^{\infty\beta}$, $\lambda_1^{\alpha\infty}$ stands for the first eigenvalue with zero Dirichlet boundary condition at $x = 0$, $x = 1$, respectively, and unchanged boundary condition at the opposite endpoint, i.e.,

$$\begin{aligned}
\lambda_1^{\alpha\beta}: &\quad (\alpha u + pu')|_{x=0} = 0, \quad (\beta u + pu')|_{x=1} = 0,\\
\lambda_1^{\infty\beta}: &\quad u(0) = 0, \quad (\beta u + pu')|_{x=1} = 0,\\
\lambda_1^{\alpha\infty}: &\quad (\alpha u + pu')|_{x=0} = 0, \quad u(1) = 0.
\end{aligned}$$

The corresponding eigenfunctions are denoted by $u_1^{\alpha\beta}$, $u_1^{\infty\beta}$ and $u_1^{\alpha\infty}$.

Theorem 1. *(i) (UAMP) holds for* (2) *if* $\lambda \in (\lambda_1^{\alpha\beta}, \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}]$ *and* $f \le 0$, $f \not\equiv 0$ *with the conclusion* $u < 0$ *in* $[0,1]$. *(ii) For every* $\lambda > \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}$ *there exists a function* $f \le 0$ *and a sign-changing solution* u *of* (2).

Remark. In [4] Clément, Peletier only considered the case $\alpha \le 0$ and $\beta \ge 0$. Our result makes no restriction on the sign of α, β.

An application of (UAMP) to multiple solutions of boundary value problems with discontinuous coefficients is given next. For a function u we use the notation $u^+ = \max\{u, 0\}$ and $u = u^+ - u^-$ so that u^+, u^- are both non-negative.

Corollary 2. *Let $f \in C[0,1]$ with $f \le 0$, $f \not\equiv 0$ and suppose $\mu < \lambda_1^{\alpha\beta}$ and $\lambda_1^{\alpha\beta} < \nu \le \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}$. Then the boundary value problem*

$$(pu')' - qu + w(\mu u^+ - \nu u^-) = f \text{ in } (0,1), \quad Bu = 0$$

has at least two solutions.

Our second result establishes an analogous theorem for the radially symmetric p-Laplacian. Recall the definition $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$, $p > 1$, where $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. The problem analogous to (1) is

$$\Delta_p v - q|v|^{p-2}v + \lambda w|v|^{p-2}v = f \text{ in } \Omega,\ |\partial_\nu v|^{p-2}\partial_\nu v + b|v|^{p-2}v = 0 \text{ on } \partial\Omega.$$

For radially-symmetric functions $v(x) = u(r)$ with $r = |x|$ on a ball $\Omega = B_1(0)$ we find $\Delta_p v(x) = L_p u(r) = r^{1-n}(r^{n-1}|u'|^{p-2}u')'$. We refer to L_p as the radially symmetric p-Laplacian. Then the boundary value problem corresponding to (2) is given by

$$L_p u - q|u|^{p-2}u + \lambda w|u|^{p-2}u = f \text{ in } (0,1), \quad B_p u = 0, \tag{3}$$

where $B_p u = 0$ is an abbreviation for the boundary condition

$$u'(0) = 0, \quad (\beta|u|^{p-2}u + r^{n-1}|u'|^{p-2}u')|_{r=1} = 0.$$

Solutions of (3) are understood such that $u, r^{n-1}|u'|^{p-2}u' \in C[0,1] \cap C^1[0,1]$. The nature of the problem is very different in the two cases $1 \le n < p$ and $n \ge p$. We restrict attention to $1 \le n < p$. In this case (3) together with arbitrary homogenous boundary data at $r = 0$ and $r = 1$ is a well defined boundary value problem, cf. Reichel, Walter [11].

Our result for (3) is restricted to the case $\beta = 0$. This has only technical reasons, cf. Lemma 8. We expect the result to hold for arbitrary $\beta \in \mathbb{R}$. We use various first eigenvalues of the Sturm-Liouville problem related to (3). The existence of theses eigenvalues was shown in [11].

$$\begin{aligned} \lambda_1^{NN}: &\quad u'(0) = 0, \quad u'(1) = 0, \\ \lambda_1^{DN}: &\quad u(0) = 0, \quad u'(1) = 0, \\ \lambda_1^{ND}: &\quad u'(0) = 0, \quad u(1) = 0. \end{aligned}$$

Theorem 3. *Suppose $1 \le n < p$ and $\beta = 0$. (i) (UAMP) holds for (3) in the class of functions $f < 0$ in $[0,1]$ if $\lambda \in (\lambda_1^{NN}, \min\{\lambda_1^{DN}, \lambda_1^{ND}\}]$ with the* *conclusion* $u < 0$ *in* $[0,1]$. *(ii) For every* $\lambda > \min\{\lambda_1^{DN}, \lambda_1^{ND}\}$ *there exists a function* $f \le 0$ *and a sign-changing solution* u *of* (3).

Remarks. (a) The restriction to $f < 0$ in $[0,1]$ is again technical, cf. Lemma 8. We expect the result to hold for $f \le 0$, $f \not\equiv 0$ as in Theorem 1.
(b) The same result holds on an annulus $A : \{x \in \mathbb{R}^n : R_1 < |x| < R_2\}$ with boundary conditions $u'(R_1) = u'(R_2) = 0$.

Corollary 4. *Let* $f \in C[0,1]$ *with* $f < 0$ *in* $[0,1]$ *and suppose* $\mu < \lambda_1^{NN}$ *and* $\lambda_1^{NN} < \nu \le \min\{\lambda_1^{ND}, \lambda_1^{DN}\}$. *Then the boundary value problem*

$$L_p u - q|u|^{p-2}u + w|u|^{p-2}(\mu u^+ - \nu u^-) = f \text{ in } (0,1), \quad u'(0) = u'(1) = 0.$$

has at least two solutions.

Arias et al. established in [1] sharp intervals for the (UAMP) for the p-Laplacian problem $\Delta_p u + \lambda |u|^{p-2}u = f(x)$ in Ω with zero Neumann boundary data on an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ with $1 \le n < p$. They showed that the interval $(\lambda_1, \bar\lambda]$ with

$$\bar\lambda = \inf_{u \in W^{1,p}(\Omega)} \int_\Omega |\nabla u|^p \, dx : \int_\Omega |u|^p \, dx = 1, \ u \text{ vanishes somewhere in } \Omega, \qquad (4)$$

is the sharp interval for (UAMP). This is consistent with our results of Theorem 1 and Theorem 3. In fact, Arias et al. show that the minimizer in (4) has exactly one zero in $\overline{\Omega}$. If compared with our results for ode-operators, one is led to conjecture that the minimizer in (4) attains its zero on $\partial\Omega$. In our case $\bar\lambda$ then coincides with an eigenvalue with a Dirichlet boundary condition at one endpoint. It remains open how the sharp result of Arias et al. can be generalized to the p-Laplacian Neumann boundary value problem $\Delta_p u - q|u|^{p-2}u + \lambda w|u|^{p-2}u = 0$ with a potential q.

The proof of (AMP) and (UAMP) given by Clément, Peletier [4] is functional analytic; the one by Arias et al. [1] is variational. On the other hand, the standard proofs of (MP) involve pointwise differential inequalities and are not functional analytic. In this paper we prove the sharp (UAMP) again with the help of differential inequalities.

2. Proof of Theorem 1 and Corollary 2

The first lemma reduces the proof of (UAMP) to a more special situation.

Lemma 5. *To prove the (UAMP) of Theorem* 1 *it suffices to prove the following weaker version: if* $f \in C[0,1]$ *with* $f < 0$ *in* $[0,1]$ *is such that the solution* u *of* (2) *has at most simple isolated zeros then* $u \le 0$ *in* $[0,1]$.

Proof. Part 1: Suppose (UAMP) of Theorem 1 holds for all $f < 0$ in $[0,1]$ with the (weaker) conclusion $u \leq 0$. We show how the full version of (UAMP) follows. Let $I = (\lambda_1^{\alpha\beta}, \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}]$ and suppose $\lambda \in I$. Since no other eigenvalue $\lambda_i^{\alpha\beta}$ lies in I we can consider the Green-function $G(x,y)$ for the boundary value problem (2), i.e.,

$$(pG'(x,y))' - qG(x,y) + \lambda wG(x,y) = -\delta_y(x) \quad \forall x, y \in (0,1) \text{ with } x \neq y,$$

where differentiation is with respect to x. Moreover $G(x,y)$ as a function of x satisfies the boundary conditions for every fixed y. By approximation of $\delta_y(x)$ by smooth strictly positive functions and by application of the hypotheses of part 1 we find that $G(x,y) \geq 0$. And since $-G(x,y)$ satisfies a linear differential equation except for $x = y$ we find $-G(x,y) > 0$ for all $x, y \in [0,1]$, $x \neq y$. Once this is established the full version of (UAMP) follows.

Part 2: Now we want to show that it even suffices to consider only those $f < 0$ in $[0,1]$, where the solution u has at most simple isolated zeros. Suppose we know that (UAMP) with conclusion $u \leq 0$ holds for all $f < 0$ in $[0,1]$ such that u has at most simple isolated zeros. Fix now an arbitrary $\tilde{f} \in C[0,1]$ with $\tilde{f} < 0$ in $[0,1]$ and let $\tilde{u}$ be the corresponding solution. Clearly $\tilde{u}$ has isolated zeros. We need to show $\tilde{u} \leq 0$. Suppose a function $\psi \in C^2[0,1]$ exists with $\psi > 0$ in $(0,1)$, $B\psi = 0$ and such that $\tilde{u}_\epsilon = \tilde{u} + \epsilon\psi$ has at most simple isolated zeros for all $\epsilon \in (0, \epsilon_0)$. Let $-\delta = \max_{[0,1]} \tilde{f} < 0$. Then

$$(p\tilde{u}_\epsilon')' - q\tilde{u}_\epsilon + \lambda w\tilde{u}_\epsilon = \tilde{f} + \epsilon\Big((p\psi')' - q\psi + \lambda w\psi\Big) \leq -\delta/2$$

provided ϵ is small enough. By the hypotheses of part 2 we obtain $\tilde{u}_\epsilon \leq 0$ and by taking the limit $\epsilon \to 0$ we also find $\tilde{u} \leq 0$. Thus, the (UAMP) with conclusion $\tilde{u} \leq 0$ holds for all $\tilde{f}$ with $\tilde{f} < 0$ in $[0,1]$.

Part 3: It remains to construct the function ψ used in part 2. If $\tilde{u}$ has a double zero at $x = 0$ or $x = 1$ then $\tilde{u} + \epsilon u_1^{\alpha\beta}$ has no zero at $x = 0, 1$. Suppose next that $\tilde{u}$ has multiple zeros at $0 < x_1 < \ldots < x_k < 1$. For small enough η we can achieve that in the interval $(x_i - \eta, x_i + \eta)$ the function $\tilde{u}'$ only vanishes at x_i, since x_i cannot be an accumulation point of zeros of $\tilde{u}'$ due to the assumption $\tilde{f} \leq -\delta < 0$. Next we choose the C^2-function ψ as shown in Figure 2 with support in $[x_i - \eta, x_i + \eta]$. If ϵ_0 is so small that $\tilde{u} + \epsilon\psi \neq 0$ in $[x_i - \eta, x_i - \frac{\eta}{2}]$ and $[x_i + \frac{\eta}{2}, x_i + \eta]$ for $\epsilon \in (0, \epsilon_0)$ then $\tilde{u} + \epsilon\psi$ has only simple zeros in $[x_i - \eta, x_i + \eta]$. This finishes the construction of ψ. □

The following transformation is standard for Sturm-Liouville eigenvalue problems. For non-zero right hand sides the proof can be adapted from Coddington, Levinson [3].

Lemma 6 (Prüfer transformation)**.** *Let u be a solution of* (2) *with at most simple zeros. Then there are C^1-functions $\rho, \phi : [0,1] \to \mathbb{R}$ with*

$$pu' = \rho\cos\phi, \quad u = \rho\sin\phi$$

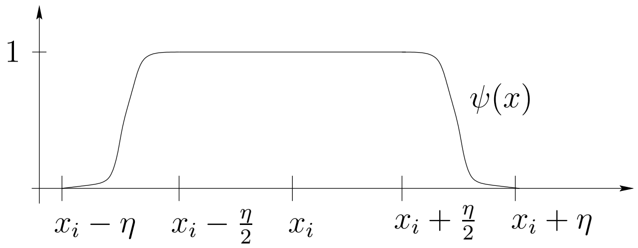

Figure 1. Choice of ψ

and $\phi(0) \in (0, 2\pi)$ *with* $\cot\phi(0) = -\alpha$, $\cot\phi(1) = -\beta$. *Moreover,* $\rho > 0$ *in* $[0,1]$ *, and* ρ, ϕ *satisfy*

$$\phi' = (-q + \lambda w)\sin^2\phi + \frac{1}{p}\cos^2\phi - \frac{f}{\rho}\sin\phi, \tag{5}$$

$$\rho' = \rho\Big((q - \lambda w + \frac{1}{p})\sin\phi\cos\phi\Big) + f\cos\phi. \tag{6}$$

Remark. Equation (5) shows that ϕ can cross the lines $k\pi$ only from below with positive slope.

Likewise, the eigenfunctions $u_1^{\alpha\infty}, u_1^{\infty\beta}$ and $u_1^{\alpha\beta}$ can be written in polar-coordinates $(\phi^{\alpha\infty}, \rho^{\alpha\infty}), (\phi^{\infty\beta}, \rho^{\infty\beta})$ and $(\phi^{\alpha\beta}, \rho^{\alpha\beta})$, where we take $u_1^{\alpha\infty}, u_1^{\beta\infty}$ **negative** but $u_1^{\alpha\beta}$ **positive**. Hence the angle-functions satisfy

$$\begin{aligned}
\phi^{\alpha\beta\prime} &= (-q + \lambda_1^{\alpha\beta} w)\sin^2\phi^{\alpha\beta} + \frac{1}{p}\cos^2\phi^{\alpha\beta}, &(7)\\
&\phi^{\alpha\beta}(0) = \pi - \operatorname{arccot}\alpha, \quad \phi^{\alpha\beta}(1) = \pi - \operatorname{arccot}\beta, &\\
\phi^{\alpha\infty\prime} &= (-q + \lambda_1^{\alpha\infty} w)\sin^2\phi^{\alpha\infty} + \frac{1}{p}\cos^2\phi^{\alpha\infty}, &(8)\\
&\phi^{\alpha\infty}(0) = 2\pi - \operatorname{arccot}\alpha, \quad \phi^{\alpha\infty}(1) = 2\pi, &\\
\phi^{\infty\beta\prime} &= (-q + \lambda_1^{\infty\beta} w)\sin^2\phi^{\infty\beta} + \frac{1}{p}\cos^2\phi^{\infty\beta}, &(9)\\
&\phi^{\infty\beta}(0) = \pi, \quad \phi^{\infty\beta}(1) = 2\pi - \operatorname{arccot}\beta. &
\end{aligned}$$

Lemma 7 (Comparison principle**).** *Assume* $g(x,s)$ *is defined on the set* $[0,1]\times\mathbb{R}$ *and is uniformly Lipschitz-continuous with respect to* s *on compact subsets of* $[0,1]\times\mathbb{R}$ *. If the functions* ϕ, ψ *are* C^1 *-functions on* $(0,1]$ *, continuous in* $[0,1]$ *with* $\phi(0) \le \psi(0)$ *and*

$$\phi' - g(x,\phi) \le 0, \quad \psi' - g(x,\psi) \ge 0 \text{ in } (0,1)$$

then the conclusion $\phi(x) \le \psi(x)$ *in* $[0,1]$ *holds. Moreover, either* $\phi < \psi$ *in* $[0,1]$ *or* $\phi \equiv \psi$ *in* $[0,1]$ *or there exists* $x_0 \in (0,1)$ *such that* $\phi = \psi$ *on* $[0, x_0]$ *and*

$\phi < \psi$ *on* $(x_0, 1]$. *The function* ϕ, ψ *is called a sub-, supersolution, respectively.*

Remark. For a pair of sub-, supersolutions ϕ, ψ with $\phi - g(x,\phi) \le 0, \not\equiv 0$ the comparison principle implies $\phi(1) < \psi(1)$. This will be used frequently in the proof of Theorem 1.

Proof. Part 1: On a finite interval $[0,1]$ the functions ϕ, ψ attain their values in the interval $[-M, M]$. Let L be the Lipschitz constant of g w.r.t. the second variable on the compact set $[0,1] \times [-M, M]$. The difference $\xi = \psi - \phi$ satisfies $\xi' \ge g(x,\psi) - g(x,\phi) \ge -L|\xi|$ on $[0,1]$. This shows that ξe^{-Lx} is increasing on intervals where ξ is negative. Since $\xi(0) \ge 0$ we get $\xi \ge 0$ on $[0,1]$.

Part 2: Now that we know $\xi = \psi - \phi \ge 0$ we find that $\xi' \ge -L\xi$ on $[0,1]$, i.e. ξe^{Lx} is increasing. In particular, if ξ is positive somewhere, then it stays positive. □

Proof of Theorem 1. First we assume $\lambda \in (\lambda_1^{\alpha\beta}, \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}]$ and show that the weaker version of (UAMP) from Lemma 5 holds. Let $f < 0$ in $[0,1]$ and let u be a solution of (2) with at most simple zeros. Let ϕ be the angle-function of u from Lemma 6. The form of the boundary condition and the assumption of simple zeros excludes the case $u(0) = 0$. Thus we are left with the two cases $\phi(0) \in (0,\pi)$ or $\phi(0) \in (\pi, 2\pi)$.

Case 1: $\phi(0) = \pi - \operatorname{arccot}\alpha \in (0,\pi)$. As long as ϕ attains values in $[0,\pi]$ we have $\phi' \ge (-q + \lambda_1^{\alpha\beta} w)\sin^2\phi + \frac{1}{p}\cos^2\phi$. Hence ϕ is a supersolution to $\phi^{\alpha\beta}$. If ϕ stayed in $[0,\pi]$ then Lemma 7 would imply $\pi \ge \phi(1) > \phi^{\alpha\beta}(1)$, i.e. ϕ could not attain the correct boundary condition. Hence ϕ must attain values in $[\pi, 2\pi]$, i.e. $\phi(x) > \pi$ for $x > x_0$. As long as ϕ attains values in $[\pi, 2\pi]$ we have $\phi' \le (-q + \lambda_1^{\infty\beta} w)\sin^2\phi + \frac{1}{p}\cos^2\phi$, i.e. ϕ is a subsolution to $\phi_1^{\infty\beta}$. Hence Lemma 7 applies on $[x_0, 1]$ and shows that ϕ stays below $\phi^{\infty\beta}$. In particular $\pi \le \phi(1) < 2\pi - \operatorname{arccot}\beta$. Thus ϕ cannot attain the prescribed boundary condition. This contradiction shows that case 1 cannot occur. The situation is depicted in Figure 2.

Case 2: $\phi(0) = 2\pi - \operatorname{arccot}\alpha \in (\pi, 2\pi)$. Clearly $\phi(x)$ stays above π. As long as $\phi \in [\pi, 2\pi]$ we have $\phi' \le (-q + \lambda_1^{\alpha\infty} w)\sin^2\phi + \frac{1}{p}\cos^2\phi$, i.e. ϕ is a subsolution to $\phi_1^{\alpha\infty}$. By Lemma 7 we get $\pi \le \phi \le \phi_1^{\alpha\infty}$, i.e, ϕ stays in $[\pi, 2\pi]$ which implies $u \le 0$ as claimed. This situation is depicted in Figure 3.

It remains to show that the interval $(\lambda_1^{\alpha\beta}, \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}]$ is the largest possible interval for the (UAMP). We use a result about the following boundary value problem:

$$(pu')' - qu + w(\mu u^+ - \nu u^-) = 0 \text{ in } (0,1), \quad Bu = 0.$$

A pair of values (μ, ν) is called a Fučik-eigenvalue if the above problem has a non-trivial solution. The set of all Fučik-eigenvalues is called the Fučik-spectrum.

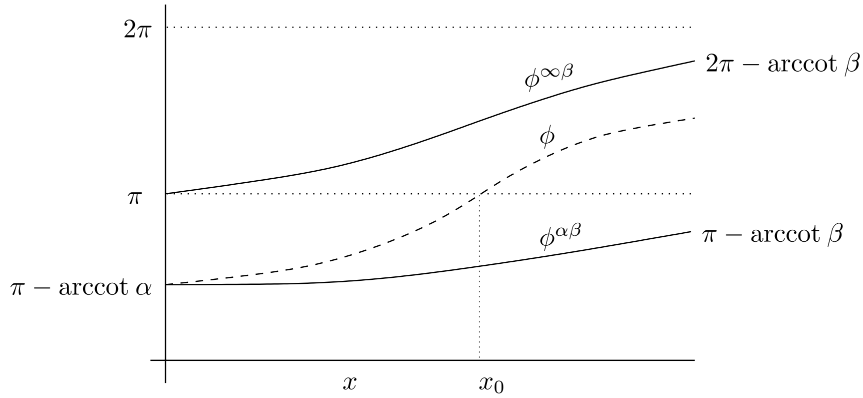

Figure 2. $\phi^{\alpha\beta}$ pushes ϕ above π, $\phi^{\infty\beta}$ keeps it below $2\pi - \operatorname{arccot}\beta$

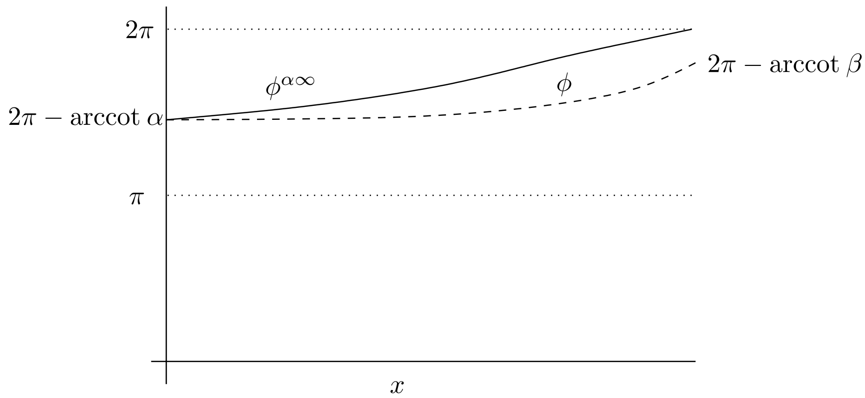

Figure 3. $\phi^{\alpha\infty}$ keeps ϕ below 2π

Next to the trivial lines $\lambda_1^{\alpha\beta} \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1^{\alpha\beta}$ the Fučik-spectrum consists of a collection of curves σ_i^+ and σ_i^-, $i = 2, \ldots, \infty$, with $\sigma_i^+ \cap \sigma_i^- = \{(\lambda_i^{\alpha\beta}, \lambda_i^{\alpha\beta})\}$. The Fučik-eigenfunctions corresponding to σ_i^+, σ_i^- have $i-1$ zeros in $(0,1)$ and are positive, negative at 0, respectively. The main result, which we are using is the asymptotic behaviour of σ_2^+, σ_2^-, as found by Reichel, Walter [11] and Rynne [12]. Let $\nu^+(\mu)$, $\nu^-(\mu)$ be the parameterizations of σ_2^+, σ_2^-. Then ν^+, ν^- are decreasing functions with the following asymptotics:

$$\begin{aligned} &\lim_{\mu\to\infty} \nu^+(\mu) = \lambda_1^{\infty\beta}, \quad \lim_{\mu \searrow \lambda_1^{\alpha\infty}} \nu^+(\mu) = \infty, \\ &\lim_{\mu\to\infty} \nu^-(\mu) = \lambda_1^{\alpha\infty}, \quad \lim_{\mu \searrow \lambda_1^{\infty\beta}} \nu^-(\mu) = \infty. \end{aligned}$$

Let $\lambda > \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}$. If, e.g., $\lambda > \lambda_1^{\infty\beta}$ then let (μ, ν) be a point on σ_2^+ with $\mu > \lambda$ and $\nu < \lambda$. Such a point exists if μ is sufficiently large. Let u be the corresponding Fučik-eigenfunction. Necessarily u is sign-changing and satisfies

$$(pu')' - qu + \lambda u = (\lambda - \mu)u^+ - (\lambda - \nu)u^- =: f \leq 0.$$

Thus (UAMP) does not hold for such a λ. If $\lambda > \lambda_1^{\alpha\infty}$ then let (μ, ν) be a point on σ_2^- with $\mu > \lambda$ and $\nu < \lambda$, which exists for sufficiently large μ. With $f \leq 0$ constructed as above we see the (UAMP) cannot hold for such a λ. ☐

Proof of Corollary 2. The proof is very simple. Let $\mu < \lambda_1^{\alpha\beta}$ and let $f \leq 0$, $f \not\equiv 0$. Since μ is not an eigenvalue the problem

$$(pu')' - qu + \mu wu = f \text{ in } (0,1), \quad Bu = 0$$

has a unique solution u, which is positive by the maximum principle (MP). Hence it also solves

$$(pu')' - qu + w(\mu u^+ - \nu u^-) = f \text{ in } (0,1), \quad Bu = 0. \tag{10}$$

If $\lambda_1^{\alpha\beta} < \nu \leq \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}$ then for the same reason

$$(pv')' - qv + \nu wv = f \text{ in } (0,1), \quad Bv = 0$$

has a unique solution v, which is negative by the (UAMP) of Theorem 1. Hence it also solves (10). ☐

3. Proof of Theorem 3 and Corollary 4

We rewrite equation (3) as

$$(r^{n-1}|u'|^{p-2}u')' - r^{n-1}q|u|^{p-2}u + \lambda r^{n-1}w|u|^{p-2}u = r^{n-1}f \text{ in } (0,1).$$

As before we can reduce the proof of (UAMP) to a simpler situation. However, since there is no p-Laplacian Green function, we need to argue differently.

Lemma 8. *To prove the (UAMP) of Theorem* 3 *is suffices to prove the following weaker version: if $f \in C[0,1]$ with $f < 0$ in $[0,1]$ is such that the solution u of* (3) *has at most simple isolated zeros then $u \leq 0$ in $[0,1]$.*

Remarks. If u satisfies boundary conditions more general than zero Neumann at $r = 0$ and $r = 1$ then we do not know how to reduce the (UAMP) to the case where u has at most simple zeros, cf. part 2 of the proof. Also, we do not know how to relax $f < 0$ to $f \leq 0$.

Proof. Part 1: Suppose (UAMP) holds for $f < 0$ in $[0,1]$ but with the weaker conclusion $u \leq 0$ in $[0,1]$. So let $\lambda \in (\lambda_1^{NN}, \min\{\lambda_1^{ND}, \lambda_1^{DN}\}]$. We want to show that $u \leq 0$ can be strengthened to $u < 0$ in $[0,1]$. Suppose $u \leq 0$ has an interior zero at $r_0 \in (0,1)$ and suppose $f \not\equiv 0$ on $[0, r_0]$ (a similar proof holds if

$f \not\equiv 0$ on $[r_0, 1]$). Then $\lambda > \int_0^{r_0}(-L_p u + q|u|^{p-2}u)ur^{n-1}\,dr / \int_0^{r_0} w|u|^p r^{n-1}\,dr$. And since $u(r_0) = 0$ we find that

$$\lambda > \frac{\int_0^{r_0}(|u'|^p + q|u|^p)r^{n-1}\,dr}{\int_0^{r_0} w|u|^p r^{n-1}\,dr} \geq \lambda_1^{ND}[0, r_0]$$

due to the variational characterization of $\lambda_1^{ND}[0, r_0]$. Since $\lambda_1^{ND}[0, r_0]$ is strictly decreasing in r_0 , we find $\lambda > \lambda_1^{ND}[0, r_0] > \lambda_1^{ND}$. This contradiction shows that u cannot have a zero in $(0, 1)$. A similar argument shows that u cannot have a zero at $r = 0$ or $r = 1$.

Part 2: Now we show that it suffices to consider those $f < 0$ in $[0, 1]$ such that the solution u has at most simple isolated zeros. Suppose we know that (UAMP) with conclusion $u \leq 0$ holds for all $f < 0$ in $[0, 1]$ such that u has at most simple isolated zeros. Fix now an arbitrary $\tilde{f} \in C[0, 1]$ with $\tilde{f} < 0$ in $[0, 1]$ and let $\tilde{u}$ be a corresponding solution. Clearly $\tilde{u}$ has isolated zeros. We need to show $\tilde{u} \leq 0$. For sufficiently small $\epsilon > 0$ the function $\tilde{u}_\epsilon = \tilde{u} + \epsilon$ has simple zeros, attains Neumann boundary-conditions at $r = 0$ and $r = 1$ and satisfies

$$(r^{n-1}|\tilde{u}_\epsilon'|^{p-2}\tilde{u}_\epsilon')' - r^{n-1}q|\tilde{u}_\epsilon|^{p-2}\tilde{u}_\epsilon + \lambda r^{n-1}w|\tilde{u}_\epsilon|^{p-2}\tilde{u}_\epsilon = r^{n-1}\tilde{f}_\epsilon,$$

where $\tilde{f}_\epsilon \to \tilde{f}$ uniformly as $\epsilon \to 0$, i.e., for sufficiently small ϵ we have $\tilde{f}_\epsilon < 0$ in $[0, 1]$. By the hypotheses of part 2 we obtain $\tilde{u}_\epsilon \leq 0$ and by taking the limit $\epsilon \to 0$ we also find $\tilde{u} \leq 0$. Thus, the (UAMP) with conclusion $\tilde{u} \leq 0$ holds for $\tilde{f} < 0$ in $[0, 1]$. □

For a Prüfer-type transformation like in Section 2 we need to suitably generalize the concept of the sine-function. Generalized sine-functions are well studied in the literature, see Lindqvist [10]. The generalized sine-function $\sin_p$ is first defined locally as the solution of the differential equation

$$u'^p + \frac{u^p}{p-1} = 1 \text{ with } u(0) = 0, u'(0) = 1. \tag{11}$$

Equation (11) arises as a first integral of $(u'^{(p-1)})' + u^{(p-1)} = 0$. The solution defines the functions $S_p(\phi) = \sin_p(\phi)$ as long as it is increasing, i.e. for $\phi \in [0, \pi_p/2]$, where

$$\frac{\pi_p}{2} = \int_0^{(p-1)^{1/p}} \frac{dt}{1 - t^p/(p-1))^{1/p}} = \frac{(p-1)^{1/p}}{p\sin(\pi/p)}\pi. \tag{12}$$

Since $S_p'(\pi_p/2) = 0$ we define S_p on the interval $[\pi_p/2, \pi_p]$ by $S_p(\phi) = S_p(\pi_p - \phi)$, and for $\phi \in (\pi_p, 2\pi_p]$ we put $S_p(\phi) = -S_p(2\pi_p - \phi)$ and extend S_p as a $2\pi_p$ -periodic function on $\mathbb{R}$. In the special case $p = 2$, $S_2(x) = \sin x$ and $\pi_2 = \pi$. The following properties of S_p will be frequently used:

Lemma 9. *For $p > 1$ the generalized sine-functions S_p have the properties:*

(i) S_p , $S_p'^{(p-1)}$ *are C^1 -functions on $\mathbb{R}$,*

(ii) S_p *solves* $|S_p'|^p + \frac{|S_p|^p}{p-1} = 1$ *on* $\mathbb{R}$,

(iii) *For* $1 < p \le 2$ *the function* S_p' *is* C^1 , *whereas for* $p \ge 2$ *the function* S_p' *is* $1/(p-1)$ *-Hölder continuous.*

Proofs can be obtained from the results of Lindqvist [10]. We show in Figure 4 the graphs of the function S_p for $p = 1.4, 2, 5$. As $p \to \infty$ the function S_p converges to $1 - |x - 1|$ and as $p \to 1$ it approaches 0 .

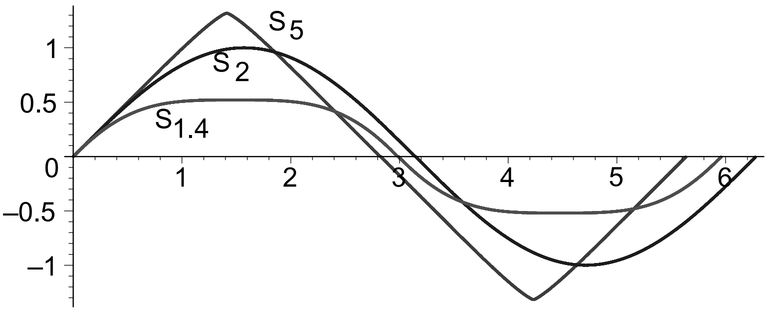

Figure 4. S_p , $p = 1.4, 2, 5$

With the help of the generalized sine-function we transform any solution of (3) with at most simple zeros into phase-space via generalized polar-coordinates ρ and ϕ . This has been done by Reichel, Walter [11] and Brown, Reichel [2] as follows:

$$r^{n-1}u'^{(p-1)} = \rho S_p'(\phi)^{(p-1)}, \qquad u^{(p-1)} = \rho S_p(\phi)^{(p-1)}. \tag{13}$$

A calculation using the defining properties of S_p and S_p' as in (ii) of Lemma 9 leads to the pair of equations:

$$\phi' = \frac{r^{n-1}}{p-1}(-q + \lambda w)|S_p(\phi)|^p + r^{\frac{1-n}{p-1}}|S_p'(\phi)|^p - \frac{r^{n-1}f}{(p-1)\rho}S_p(\phi), \tag{14}$$

$$\rho' = \rho\Big\{\Big(r^{n-1}(q - \lambda w) + r^{\frac{1-n}{p-1}}\Big)S_p(\phi)^{(p-1)}S_p'(\phi)\Big\} + r^{n-1}fS_p'(\phi). \tag{15}$$

Radially symmetric solutions of (3) satisfy $u'(0) = u'(1) = 0$. This amounts to $\phi(0) = \pi_p/2 \mod \pi_p$ and $\phi(1) = \pi_p/2 \mod \pi_p$. The following results show that eigenvalues with arbitrary homogeneous boundary conditions at $r = 0$ and $r = 1$ exist provided $1 \le n < p$.

Proposition 10 (Reichel, Walter [11]). *Let* $1 \le n < p$ *and consider the eigenvalue problem*

$$(r^{n-1}|u'|^{p-2}u')' - r^{n-1}q|u|^{p-2}u + \lambda r^{n-1}w|u|^{p-2}u = 0 \text{ in } (0,1), \tag{16}$$

with the boundary conditions

$$\begin{aligned} (\alpha_1|u|^{p-2}u + \alpha_2 r^{n-1}|u'|^{p-2}u')|_{r=0} &= 0, \\ (\beta_1|u|^{p-2}u + \beta_2 r^{n-1}|u'|^{p-2}u')|_{r=1} &= 0, \end{aligned} \tag{17}$$

where $\alpha_1^2+\alpha_2^2 > 0$ *,* $\beta_1^2+\beta_2^2 > 0$ *. It has a countable number of simples eigenvalues* $\lambda_1 < \lambda_2 < \dots$ *,* $\lim_{n\to\infty}\lambda_n = \infty$ *, and no other eigenvalues. The corresponding eigenfunction* u_n *has* $n-1$ *simple zeros in* $(0,1)$ *.*

We denote the first eigenfunction for $\alpha_1 = \beta_1 = 0$ by u_1^{NN} since it has vanishing Neumann data at both endpoints. Likewise, if $\alpha_1 = \beta_2 = 0$ we denote the first eigenfunction by u_1^{ND} since it has zero Dirichlet data at $r = 1$. Finally, the first eigenfunction corresponding to $\alpha_2 = \beta_1 = 0$ is denoted by u_1^{DN}. Each of these three functions can be written in generalized polar coordinates $(\phi^{NN}, \rho^{NN}), (\phi^{ND}, \rho^{ND})$ and (ϕ^{DN}, ρ^{DN}), where we take u_1^{ND}, u_1^{DN} **negative** but u_1^{NN} **positive**. Hence the angle-functions satisfy

$$\begin{aligned} \phi^{NN\prime} &= \frac{r^{n-1}}{p-1}(-q+\lambda_1^{NN}w)|S_p(\phi^{NN})|^p + r^{\frac{1-n}{p-1}}|S_p'(\phi^{NN})|^p, && (18) \\ & \quad \phi^{NN}(0) = \pi_p/2, \quad \phi^{NN}(1) = \pi_p/2, \\ \phi^{ND\prime} &= \frac{r^{n-1}}{p-1}(-q+\lambda_1^{ND}w)|S_p(\phi^{ND})|^p + r^{\frac{1-n}{p-1}}|S_p'(\phi^{ND})|^p, && (19) \\ & \quad \phi^{ND}(0) = 3\pi_p/2, \quad \phi^{ND}(1) = 2\pi_p, \\ \phi^{DN\prime} &= \frac{r^{n-1}}{p-1}(-q+\lambda_1^{DN}w)|S_p(\phi^{DN})|^p + r^{\frac{1-n}{p-1}}|S_p'(\phi^{DN})|^p, && (20) \\ & \quad \phi^{DN}(0) = \pi_p, \quad \phi^{DN}(1) = 3\pi_p/2. \end{aligned}$$

As a basic tool for our analysis we use the following more subtle version of the comparison principle, cf. Lemma 7. Such comparison principles can be found in detail in Walter [13].

Lemma 11 (Generalized comparison principle)**.** *Let* $g(r,s)$ *be defined on the set* $[0,1]\times\mathbb{R}$ *and suppose it satisfies a generalized local Lipschitz-condition w.r.t.* s *, i.e., for every* $M > 0$ *there exists a function* $h \in L^1(0,1)$ *such that*

$$|g(r,s_1) - g(r,s_2)| \le h(r)|s_1 - s_2| \quad \forall |s_1|, |s_2| \le M, \forall r \in (0,1).$$

If the functions ϕ, ψ *are* C^1 *-functions on* $(0,1]$ *, continuous in* $[0,1]$ *with* $\phi(0) \le \psi(0)$ *and*

$$\phi' - g(r,\phi) \le 0, \quad \psi' - g(r,\psi) \ge 0 \text{ in } (0,1)$$

then the conclusion $\phi(r) \le \psi(r)$ *in* $[0,1]$ *holds. Moreover, either* $\phi < \psi$ *in* $[0,1]$ *or* $\phi \equiv \psi$ *in* $[0,1]$ *or there exists* $r_0 \in (0,1)$ *such that* $\phi = \psi$ *on* $[0, r_0]$ *and* $\phi < \psi$ *on* $(r_0, 1]$ *. The function* ϕ *,* ψ *is called a sub-, supersolution, respectively.*

Proof. The proof is similar to Lemma 7. The function $\xi = \psi - \phi$ satisfies $\xi' \geq g(r,\psi) - g(r,\phi) \geq -h(r)|\xi|$ on intervals $[0,1]$. This shows that $\xi e^{-\int_0^r h(t)dt}$ is increasing on intervals where ξ is negative. Since $\xi(0) \geq 0$ we get $\xi \geq 0$ on $[0,1]$. Once $\xi \geq 0$ is known one obtains that $\xi' \geq -h(r)\xi$ on $[0,1]$, i.e. $\xi e^{\int_0^r h(t)dt}$ is increasing. As before we find that if ξ is positive somewhere, then it stays positive. □

Proof of Theorem 3. The proof is very similar to the proof of Theorem 1. In the ϕ-equation (14) of the Prüfer-transform the singular function $r^{\frac{1-n}{p-1}}$ appears. This function is in $L^1(0,1)$ precisely for $1 \leq n < p$. Hence we can replace the comparison principle of Lemma 7 by the generalized comparison principle of Lemma 11. Figure 2, Figure 3 used in the proof of Theorem 1 still provide a graphical insight into case 1, case 2, respectively, below. Let $f < 0$ in $[0,1]$ and let u be a solution of (3) with at most simple zeros and angle-function ϕ. We consider two cases: $\phi(0) = \pi_p/2$ or $\phi(0) = 3\pi_p/2$.

Case 1: $\phi(0) = \pi_p/2$. As long as ϕ attains values in $[0,\pi_p]$ we have $\phi' \geq \frac{r^{n-1}}{p-1}(-q + \lambda_1^{NN} w)|S_p(\phi)|^p + r^{\frac{1-n}{p-1}}|S_p'(\phi)|^p$, i.e., ϕ is a supersolution to ϕ^{NN}. Hence ϕ^{NN} pushes ϕ above π_p, i.e. $\phi(r) > \pi_p$ for $r > r_0$. As long as ϕ attains values in $[\pi_p, 2\pi_p]$ we have $\phi' \leq \frac{r^{n-1}}{p-1}(-q + \lambda_1^{DN} w)|S_p(\phi)|^p + r^{\frac{1-n}{p-1}}|S_p'(\phi)|^p$ i.e. ϕ is a subsolution to ϕ_1^{DN}. Hence $\pi_p \leq \phi(1) < 3\pi_p/2$. Thus ϕ cannot attain the prescribed boundary condition. This contradiction shows that case 1 cannot occur.

Case 2: $\phi(0) = 3\pi_p/2$. Clearly $\phi(r)$ stays above π_p. As long as $\phi \in [\pi_p, 2\pi_p]$ we have $\phi' \leq \frac{r^{n-1}}{p-1}(-q + \lambda_1^{ND} w)|S_p(\phi)|^p + r^{\frac{1-n}{p-1}}|S_p'(\phi)|^p$, i.e., ϕ is a subsolution to ϕ_1^{ND}. Hence ϕ stays in $[\pi_p, 2\pi_p]$ as claimed.

Finally we need to show that the interval $(\lambda_1^{NN}, \min\{\lambda_1^{DN}, \lambda_1^{ND}\}]$ is the largest possible interval for (UAMP). The corresponding Fučik problem is

$$L_p u - q|u|^{p-2}u + w|u|^{p-2}(\mu u^+ - \nu u^-) = 0 \in (0,1), \quad u'(0) = u'(1) = 0.$$

Again the Fučik-spectrum consists of the the trivial lines $\lambda_1^{NN} \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1^{NN}$ and a collection of curves σ_i^+ and σ_i^-, $i = 2, \ldots, \infty$ with $\sigma_i^+ \cap \sigma_i^- = \{(\lambda_i^{NN}, \lambda_i^{NN})\}$. The Fučik-eigenfunctions corresponding to σ_i^+, σ_i^- have $i-1$ zeros in $(0,1)$ and are positive, negative at 0, respectively. By the result of Reichel, Walter [11] the asymptotic behaviour of the descreasing curves σ_2^+, σ_2^- is given as follows:

$$\begin{aligned} &\lim_{\mu\to\infty} \nu^+(\mu) = \lambda_1^{DN}, \quad \lim_{\mu \searrow \lambda_1^{ND}} \nu^+(\mu) = \infty, \\ &\lim_{\mu\to\infty} \nu^-(\mu) = \lambda_1^{ND}, \quad \lim_{\mu \searrow \lambda_1^{DN}} \nu^+(\mu) = \infty. \end{aligned}$$

Just as in Theorem 1 this enables us to shows that (UAMP) cannot hold for $\lambda > \min\{\lambda_1^{\infty\beta}, \lambda_1^{\alpha\infty}\}$. □

Proof of Corollary 4. Let $\mu < \lambda_1^{NN}$ and let $f < 0$ in $[0,1]$. Then

$$L_p u - q|u|^{p-2}u + \mu w|u|^{p-2}u = f \text{ in } (0,1), \quad u'(0) = u'(1) = 0$$

has a positive solution u. In fact, u can be obtained as a minimizer of the functional $J[u] = \int_0^1 (|u'|^p + q|u|^p - \mu w|u|^p + pfu) r^{n-1}\, dr$ in $C^1[0,1]$, which is bounded below by the assumption $\mu < \lambda_1^{NN}$. Since $|u|$ also provides a minimizer, we may assume $u \geq 0$. Hence u solves

$$L_p u - q|u|^{p-2}u + w|u|^{p-2}(\mu u^+ - \nu u^-) = f \text{ in } (0,1), u'(0) = u'(1) = 0. \tag{21}$$

Likewise, if $\lambda_1^{NN} < \nu \leq \min\{\lambda_1^{DN}, \lambda_1^{ND}\}$ then

$$L_p u - q|v|^{p-2}v + \nu w|v|^{p-2}v = f \text{ in } (0,1), \quad v'(0) = v'(1) = 0$$

has a solution v, cf. Reichel, Walter [11], Theorem 3. By the (UAMP) of Theorem 3 we find $v \leq 0$. Hence v also solves (21). □

4. Examples

For three different boundary value problems we determine the optimal parameter ranges for (UAMP) explicitely/numerically.

4.1. The Fourier equation

The simplest possible problem is the Fourier-problem

$$u'' + \lambda u = f \text{ on } (0,1), \quad u'(0) = 0, u'(1) + \beta u(1) = 0.$$

We are interested in the question how the optimal parameter range for (UAMP) changes with β. The first eigenvalue $\lambda_1^{0\beta}$ is given implicitly by

$$\begin{aligned} \beta > 0: &\qquad \lambda_1^{0\beta} = \text{ first positive solution of } \beta = \sqrt{\lambda}\tan\sqrt{\lambda} \\ \beta < 0: &\qquad \lambda_1^{0\beta} = \text{ first negative solution of } \beta = \sqrt{|\lambda|}\tanh\sqrt{|\lambda|}. \end{aligned}$$

Similarly, the first eigenvalue $\lambda_1^{\infty\beta}$ with zero Dirichlet boundary data at $x = 0$ is given by

$$\begin{aligned} \beta > -1: &\qquad \lambda_1^{\infty\beta} = \text{ first positive solution of } \beta = -\sqrt{\lambda}\cot\sqrt{\lambda} \\ \beta < -1: &\qquad \lambda_1^{\infty\beta} = \text{ first negative solution of } \beta = -\sqrt{|\lambda|}\coth\sqrt{|\lambda|}, \end{aligned}$$

and $\lambda_1^{0\infty} = \pi^2/4$ is the first eigenvalue with zero Neumann at $x = 0$ and zero Dirichlet at $x = 1$. Hence, by Theorem 1 the (UAMP) holds for $\lambda_1^{0\beta} < \lambda \leq \min\{\lambda_1^{\infty\beta}, \lambda_1^{0\infty}\}$. This is shown in Figure 5.

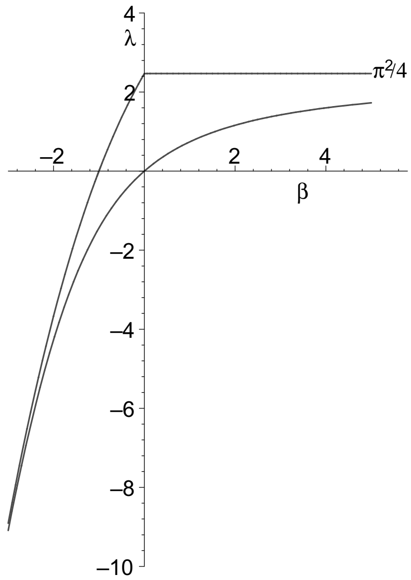

Figure 5. (UAMP) holds between the two curves

4.2. The radially symmetric Laplacian on annuli

Let $A_b = \{x \in \mathbb{R}^3 : 1 < |x| < b\}$ be a 3-dimensional annulus. We consider the boundary value problem

$$\Delta u + \lambda u = f \text{ in } A, \quad \partial_\nu u = 0 \text{ on } \partial A$$

under the assumption of radial symmetry $f = f(r), u = u(r)$. Thus, we have

$$u'' + \frac{2}{r}u' + \lambda u = f \text{ in } (1, b), \quad u'(1) = u'(b) = 0.$$

Clearly $\lambda_1^{00} = 0$ (two zero-Neumann boundary conditions). The other two eigenvalues $\lambda_1^{\infty 0}$ (Dirichlet at 1 , Neumann at b) and $\lambda_1^{0\infty}$ (Neumann at 1 , Dirichlet at b) can be determined with help of the transformation $w(x) = ((b-1)x + 1)u(b(x-1)+1)$. The eigenvalue problem then becomes

$$\begin{aligned} &w'' + \lambda(b-1)^2 w = 0 \text{ in } (0,1), \\ \lambda_1^{\infty 0}: \quad &w(0) = 0, \quad w'(1) - \tfrac{b-1}{b} w(1) = 0, \\ \lambda_1^{0\infty}: \quad &w'(0) - (b-1)w(0) = 0, \quad w(1) = 0. \end{aligned}$$

Implicitly, the eigenvalues are given by

$$\lambda_1^{\infty 0} = \frac{\mu}{(b-1)^2}: \qquad \mu = \text{ first positive solution of } \frac{b}{b-1} = \frac{\tan\sqrt{\mu}}{\sqrt{\mu}}$$

$$\lambda_1^{0\infty} = \frac{\mu}{(b-1)^2}: \qquad \mu = \text{ first positive solution of } -\frac{1}{b-1} = \frac{\tan\sqrt{\mu}}{\sqrt{\mu}}.$$

It turns out that $\lambda_1^{\infty 0} < \lambda_1^{0\infty}$. Hence, (UAMP) holds for $0 < \lambda \le \lambda_1^{\infty 0}$, as shown in Figure 6.

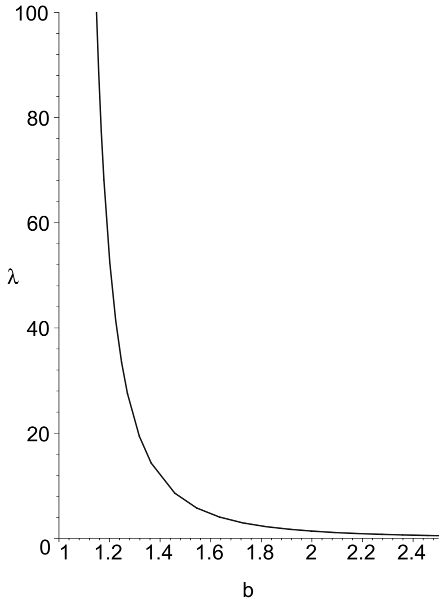

Figure 6. (UAMP) holds between $\lambda_1^{00} = 0$ and the curve

4.3. The radially symmetric p-Laplacian on balls

On the ball $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ in $\mathbb{R}^n$ we consider the radially symmetric boundary value problem

$$\Delta_p u + \lambda |u|^{p-2} u = f \text{ in } B_1(0), \quad \partial_\nu u = 0 \text{ on } \partial B_1(0),$$

i.e., we have

$$r^{n-1}(|u'|^{p-2}u')' + \lambda r^{n-1}|u|^{p-2}u = r^{n-1} f \text{ in } (0,1), \quad u'(0) = u'(1) = 0.$$

In this case $\lambda_1^{NN} = 0$. With the help of a Fortran-code described in Brown, Reichel [2] we computed the eigenvalues λ_1^{DN} and λ_1^{ND}. The results for $n = 2$ and $n = 3$ are shown in Table 1. It appears that λ_1^{DN} is smaller than λ_1^{ND}. Hence (UAMP) holds between 0 and λ_1^{DN}.

p	λ_1^{DN}	λ_1^{ND}
2.1	0.16953	6.15389
2.5	0.51507	7.71025
3	0.81843	9.83149
4	1.29817	14.68165
5	1.72125	20.34705
6	2.12243	26.82324
7	2.51267	34.10377
8	2.89664	42.18266
9	3.27666	51.05474
10	3.65402	60.71532

p	λ_1^{DN}	λ_1^{ND}
3.1	0.01023	20.32758
3.5	0.06534	25.21349
4	0.15534	32.21492
5	0.33096	49.43648
6	0.49451	71.35609
7	0.64996	98.45043
8	0.80025	131.18846
9	0.94709	170.03313
10	1.09155	215.44222

TABLE 1. $n = 2$ *(left)* $n = 3$ *(right)*

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